

HOW TO PLAY MACROSCOPIC QUANTUM GAME

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Abstract

Quantum games are usually considered as games with strategies defined not by the standard Kolmogorovian probabilistic measure but by the probability amplitude used in quantum physics. The reason for the use of the probability amplitude or "quantum probabilistic measure" is the nondistributive lattice occurring in physical situations with quantum microparticles. In our paper we give examples of getting nondistributive orthomodular lattices in some special macroscopic situations without use of quantum microparticles.

Mathematical structure of these examples is the same as that for the spin one half quantum microparticle with two non-commuting observables being measured. So we consider the so called Stern-Gerlach quantum games. In quantum physics it corresponds to the situation when two partners called Alice and Bob do experiments with two beams of particles independently measuring the spin projections of particles on two different directions. In case of coincidences defined by the payoff matrix Bob pays Alice some sum of money. Alice and Bob can prepare particles in the beam in certain independent states defined by the probability amplitude so that probabilities of different outcomes are known. Nash equilibrium for such a game can be defined and it is called the quantum Nash equilibrium.

The same lattice occurs in the example of the firefly flying in a box observed through two windows one at the bottom another at the right hand side of the box with a line in the middle of each window. This means that two such boxes with fireflies inside them imitate two beams in the Stern-Gerlach quantum game. However there is a difference due to the fact that in microscopic case Alice and Bob freely choose the representation of the lattice in terms of non-commuting projectors in some Hilbert space. In our macroscopic imitation there is a problem of the choice of this representation (of the

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angles between projections). The problem is solved by us for some special forms of the payoff matrix. We prove the theorem that quantum Nash equilibrium occurs only for the special representation of the lattice defined by the payoff matrix. This makes possible imitation of the microscopic quantum game in macroscopic situations. Other macroscopic situations based on the so called opportunistic behavior leading to the same lattice are considered.

In this paper we continue the investigation [1] of macroscopic situations described by the same mathematical formalism as some simple (spin one half and spin one) quantum systems. In these situations stochasticity is described by some wave function as vector in finite dimensional Hilbert space with non-commuting operators in it as observables.

So one has the complementarity property for such systems. These situations can occur in economics [2, 3], biology [4] etc so that chance in these sciences must not necessarily be described by the standard Kolmogorovian probability measure as it is usually supposed to be but by the more general quantum formalism. In [1] it was shown that new type of Nash equilibrium can arise in these cases. Differently from the microworld the Planck's constant does not play any role in our examples.

1. The Firefly in a Box

Here we consider some other than in our papers [1, 5, 6] example named "the firefly in a box" [8]. This example will be used by us in order to show the connection between the Boolean lattice with probabilistic description of chance and the non Boolean nondistributive lattice with the quantum probability measure on it. The example is the simplification of the well known more complex Foulis "firefly in a box" case [9, 10]. We take it because in our paper [1] we found Nash equilibrium for a quantum game based on Hasse diagram for this simplified case. It occurred that in quantum case there are Nash equilibrium more profitable than the classical ones. These Nash equilibria correspond to realizations of the nondistributive lattice in terms of certain noncommutative projectors in Hilbert space of the spin one half system. We shall not discuss other known examples (Wright urn [11]) leading to nondistributive lattices because we did not investigate Nash equilibrium for these examples. The rule for the quantum macroscopic game for our example can be formulated as follows. A firefly (surely a macroscopic agent) is roaming around a box and some observer (other macroscopic agent) can

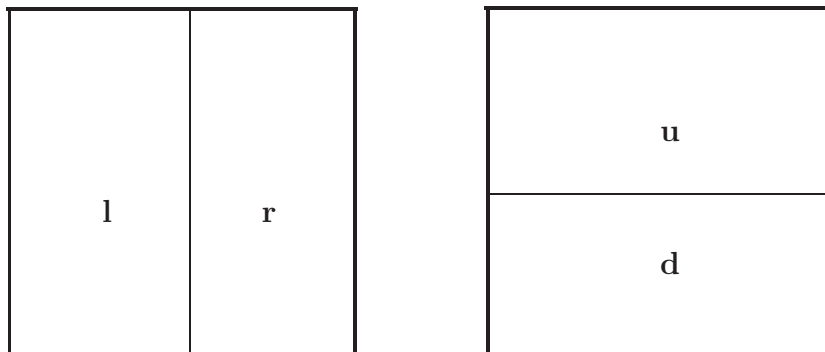


Fig. 1: The firefly in a box with two windows.

see it either through the window at the bottom of the box or through the window at the right side of it. Each window has a thin line perpendicular to it drawn at its center so that an observer can see the firefly in one or another halves of the box.

An observer cannot look at the same moment at two windows at once so that one has two incompatible experiments. Let us call possible observable situations as "left", "right", "up" and "down". The outcomes of the experiments can be described by the nondistributive lattice employed by Birkhoff and von Neumann [12] for the spin one half system with two complementary observables – (different spin projections) – being measured. This is an orthocomplemented lattice. All rigorous mathematical definitions can be found in [13]. However for better understanding of the paper we must give some necessary definitions and conjectures here.

The lattice L is a partially ordered set (S, \leq) with two operations \vee, \wedge so that each pair $x, y \in L, x \neq y$ has a supremum $x \vee y$ and an infimum $x \wedge y$. There are elements $\emptyset \in L, I \in L$ such that $x \vee \emptyset = x, x \wedge \emptyset = \emptyset, x \vee I = I, x \wedge I = x$.

The lattice is complemented if for $\forall x \in L$ exists at least one complement x' such that $x \wedge x' = \emptyset, x \vee x' = I$. The elements of the lattice are orthogonal $x \perp y$ if $x \leq y'$. The operations \wedge, \vee can sometimes be understood as logical disjunction and conjunction. Then it is supposed that if x is true then $x \vee y$ is true, if y is true then $x \vee y$ is true. However in quantum logic "if" is not equal to "always if" (iff). So in general it is not correct to think that from $x \vee y$ true follows that either of them is true.

Negation in quantum logic is realized through orthogonality. There is

some discussion in literature on the problem of logical interpretation of lattices (see [14]) for the general case so that different views arise from different definitions. For Boolean sublattices of the non Boolean lattice it is possible to give usual logical interpretation of the lattice operations.

The success of quantum physics shows that the idea of Birkhoff and von Neumann of the nondistributive lattice with quantum probabilistic measure on it as justification of use of the probability amplitude is a correct one. So it seems reasonable to conclude that in other cases when the same lattices occur one must obtain the quantum mechanical mathematical formalism. One can draw the so called Hasse diagram for the lattice so that lines correspond to partial order, going up one can obtain intersection of lines at \vee , going down one can obtain intersection at \wedge . Here "l", "r", "u", "d" are elements (logical

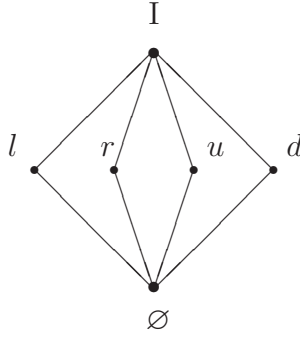


Fig. 2: Hasse diagram of the nondistributive lattice of the firefly in a box example.

atoms) of the lattice describing different experimentally testable propositions for the firefly on Fig.1. Elements "l" and "r" as well as "u" and "d" are orthogonal. One also has for the two lattice operations \wedge ("and"), \vee ("or")

$$l \vee r = u \vee d = l \vee u = r \vee d = l \vee d = r \vee u = I$$

$$l \wedge r = l \wedge u = l \wedge d = r \wedge u = u \wedge d = r \wedge d = \emptyset$$

which means that "l" or "r" is always true while "l" and "r" is always false etc. For example "r" and "u" is always false because "experimentally" there is no such observable element at the disposal of the observer due to the impossibility of simultaneous observation of the corresponding situations. The lattice is nondistributive because

$$l \wedge (r \vee d) = l \wedge I = l \neq (l \wedge r) \vee (l \wedge d) = \emptyset \vee \emptyset = \emptyset.$$

If the firefly randomly moves inside the box the observer can describe the outcomes of his observations as some representation of the nondistributive lattice in terms of projectors ("yes - no" questions) in two dimensional real space. He (she) defines the "quantum" probability of the outcomes from some wave function $|\Psi\rangle$ taken as the two dimensional vector by

$$w_\alpha = \langle \Psi | \hat{P}_\alpha | \Psi \rangle, \alpha \in \{l, r, u, d\}, \quad (1)$$

here one has

$$w_l + w_r = w_u + w_d = 1. \quad (2)$$

So one has the wave function and non-commuting operators – the projectors \hat{P}_α for the "firefly in a box" example. To organize the game first consider the game in which quantum microparticles are used. Call it the Stern-Gerlach quantum game [1]. Two partners Alice and Bob are sitting close to accelerators and prepare two beams of particles (protons) with spin one half. Then they do the Stern-Gerlach experiment measuring two different spin projections of their particles. There is some payoff matrix given by the table 1. The meaning of it is that in case Alice gets result **1** and Bob gets **3** Bob pays

Table 1: payoff of Alice

strategies	Bob			
Alice	1	2	3	4
1	0	0	c_3	0
2	0	0	0	c_4
3	c_1	0	0	0
4	0	c_2	0	0

to Alice the sum of money c_3 prescribed by the payoff matrix etc. There are some frequencies p_1 of getting **1** by Alice in a series of her measurements and there are frequencies q_3 of getting **3** by Bob. These frequencies are defined by the probabilities of certain outcomes which can be calculated by the rules of quantum physics if one knows the wave functions of particles prepared by Alice and Bob in their beams. The average profit of Alice can be calculated as

$$\begin{aligned} \overline{H}_A &= c_1 p_3 q_1 + c_3 p_1 q_3 + c_2 p_4 q_2 + c_4 p_2 q_4 \\ p_1 + p_3 &= 1, \quad p_2 + p_4 = 1, \quad q_1 + q_3 = 1, \quad q_2 + q_4 = 1 \end{aligned} \quad (3)$$

The strategies of Alice and Bob in this game are described by the wave functions of particles in their beams. It is supposed that Alice and Bob have special experimental setups to produce their particles in certain states with some fixed wave function. However they are not free in their choice of measuring spin projections. It is supposed that both partners know what projections are to be measured, For macroscopic situations described by the same lattice as the Stern-Gerlach quantum game type one can be interested to find answers on the questions: what is the meaning of the "preparation" of the wave function and what is the meaning of measuring different non-commuting observables from the point of view of our macroscopic agents?

To find the answer one must take into account the fact that this lattice can be embedded into some Boolean lattice. The physical meaning of the embedment is simple: the observer cannot check some situations described by some more general Boolean lattice due to the character of his (her) experiments. These elements of the lattice are "hidden variables" for the observer and the non-distributivity of the lattice is the payment for his (her) "ignorance". A general theory of realizing of quantum logical lattices as "concrete logics" obtained from Boolean lattices was developed in [16].

Surely due to Kochen-Specker's theorem [16] not all quantum logical lattices can be embedded into Boolean lattices, that is why quantum theory of microparticles is basic and is not a hidden variable theory. The other objection is breaking of Bell's inequalities for relativistic systems if entangled states are considered [17].

However for our aim to find macroscopic situations with macroscopic agents with behavior described by the quantum rules (i.e. by the Born-Luders rule for calculation of probabilities) the class of such "quasi-classical" lattices [5, 8] embedded in Boolean ones is wide enough. To construct the Boolean lattice divide the box on four parts.

The firefly can occur in any of the four parts. Construct the Boolean lattice based on elements 1, 2, 3, 4 as it's atoms. One obtains the nondistributive lattice of the Fig.2 from the distributive lattice by considering only composite elements "l", "r", "u", "d" of the second row of the boolean lattice while the atomic elements 1, 2, 3, 4 as well as the third row composed from triples are unobservable.

So the main lesson is that one can expect obtaining of the quantum rules in situations when one has stochastic processes which are secondary to some basic non observable ones. This is typical for situations on the stock market, in some complex biological systems etc. The other important

w_2	w_4
w_1	w_3

Fig. 3: Probability of different firefly in a box positions from it's "own" point of view.

feature is complementarity as impossibility of checking all properties at the same moment of time. What is the connection of the probabilities on the Boolean lattice and the quantum probability on the nondistributive one?

Denoting $w_a : w_1, w_2, w_3, w_4$ the probabilities for atoms of the Boolean lattice one obtains the equations

$$\begin{aligned}
w_1 + w_2 &= w_l \\
w_3 + w_4 &= 1 - w_l \\
w_1 + w_3 &= w_d \\
w_2 + w_4 &= 1 - w_d
\end{aligned} \tag{4}$$

From these equations it is easy to see that the "strange" quantum rule for one and the same object (the firefly) is not strange at all if it is considered not for the elementary events but for the complex ones!

$$w_l + w_r + w_u + w_d = 2 \tag{5}$$

It seems that any distribution w_1, w_2, w_3, w_4 leads to some fixed wave function and some fixed representation of the lattice in terms of projectors on some fixed directions on the plane. However one can see that distributions leading to appearance of two ones or zeros for complex events of the second floor are prohibited due to the definition of disjunction in the quantum logical lattice while it is possible in the Boolean case. This means that logical atoms of the first floor cannot be definitely determined from the point of the observer for the quantum logical case being totally "hidden" for him (her). One can ask the question: what prohibits the firefly to occur in the corner **1**?

Is it possible for the observer to get **1** for the outcome "left" and **1** for the outcome "down"? The answer is surely positive. But for the quantum system it is impossible to get eigenfunction for non-commuting spin operators. Is it a contradiction?

The answer is "no"! It is here where something like the wave packet collapse idea comes into the play. Quantum theory does not forbid to get in observations of complementary observables positive results. It only says that if one prepares the wavefunction as an "eigenfunction of the projector on the "left" part and performs many observations of complementary observables then one obtains some probability distribution for the complementary observable "up"- "down". This distribution is not 1, 0 because the firefly has some freedom and the only instruction for him given on the preparation stage is to be "somewhere" in the "left" part. The observer cannot always let him be in the corner¹, so in many experiments with the same "left" instruction the results will be sometimes "up", sometimes "down" with frequencies obtained from the wave function and the representation of observables in the form of operators.

If the distributions w_l, w_d are known the distribution w_1, w_2, w_3, w_4 obtained from it is not unique: it is defined up to some arbitrary w_i where w_i is some probability for fixed i .

This is just the manifestation of "indefiniteness" of the quantum situation in comparison to Boolean one as we just said before.

2. Stern-Gerlach quantum game

The macroscopic quantum game considered previously in [1] called the macroscopic Stern-Gerlach quantum game can be organized for the "firefly in a box" case as follows.

There are two partners Alice and Bob and two boxes with fireflies there. Alice can try to choose some classical probability distribution for Boolean elementary outcomes 1, 2, 3, 4 with limitations discussed before. This can be done by "training" of the firefly stimulating it to come more often to this or that part of the box. For example any observation by Alice of the part of the box is accompanied by the flash of light with some prolongation in time stimulating the firefly to react. Different times of observation result in different frequencies for the firefly to occur in some part of the box. Supposing that Alice is interested only in l, r, u, d outcomes she cannot define exact

distributions for Boolean elementary outcomes but only some class of it. However from the quantum point of view definition of frequencies for l, r, u, d means definition of the wave function and the representation of the atoms of the non Boolean lattice in terms of non-commuting projectors, i. e. definition of the angle between different spin projections. This can be called "the preparation stage".

In special cases getting 1, 0 for some of the complementary possibilities she can speak about some fixed wave function as an eigenfunction of some spin projection operator. However in general case it is not the case.

Same manipulations are made by Bob. However neither Alice nor Bob have knowledge of the training procedures of the partner.

Then the game begins. Alice and Bob with their trained fireflies look at the results of their observations accompanied by flashes of light. In cases defined by the rules of the game when for example Alice gets some fixed result " α " while Bob gets " β " Bob must pay money to Alice etc.

There is some payoff matrix defining in what cases Bob pays Alice some money as it was defined in [1]. The profit depends on the frequencies of the outcomes. The average profit is calculated by using the quantum rule for projectors \hat{P}_α^a for Alice, \hat{P}_α^b for Bob

$$\begin{aligned} \overline{H} &= \langle \Psi_A | \langle \Psi_B | \hat{H} | \Psi_B \rangle | \Psi_A \rangle; \\ \hat{H} &= c_3 \hat{P}_u^a \otimes \hat{P}_d^b + c_1 \hat{P}_d^a \otimes \hat{P}_u^b + c_4 \hat{P}_l^a \otimes \hat{P}_r^b + c_2 \hat{P}_r^a \otimes \hat{P}_l^b. \end{aligned} \quad (6)$$

which in terms of the "Boolean philosophy" means calculation of

$$\overline{H} = c_3 w_u w_d^b + c_1 w_d w_u^b + c_4 w_l w_r^b + c_2 w_r w_l^b, \quad (7)$$

$$w_u + w_d = w_l + w_r = w_u^b + w_d^b = w_l^b + w_r^b = 1$$

where w_α, w_α^b are probabilities used by Alice and Bob. One can recognize in (7) formula (3) with w_α^b playing the role of q_α, p_α .

Here one must make some remarks about some peculiar features of this calculation. The representation of the non Boolean lattice in terms of non-commuting projectors is not unique. It is defined up to some angles θ, τ . Existence of different representations of the non Boolean lattice parameterized by the angles is the manifestation of the freedom of the observer to choose measurement of any spin observable for the real quantum system. There is also another freedom for the observer manifested in the choice of the wave function.

Let us defined

$$\begin{aligned} w_u &= \cos^2 \alpha, & w_l &= \cos^2(\alpha - \theta) \\ w_u^b &= \cos^2 \beta, & w_r^b &= \cos^2(\beta - \tau) \end{aligned}$$

If after this definition of the angle Alice and Bob cannot change the angles then their freedom is now limited by choosing only some special distributions for the firefly in the complementary experiment. The "quantum logic" leads to arising of a special "quantum correlation" between complementary observations. This correlation according to [1] can be expressed by the constraint

$$\frac{(w_u + w_l - 1)^2}{\cos^2 \theta} + \frac{(w_l - w_u)^2}{\sin^2 \theta} = 1. \quad (8)$$

Different choice of the angles leads to the different constraint. Different Nash equilibrium can be found for different angles. It was shown in [1] for the Hasse diagram considered in this paper that some of these equilibrium are more profitable for partners others are less. So one can put the hypothesis that it is the more profitable equilibrium that can play the role of the principle of choice of the representation of the lattice.

3. Eigenequilibrium

There is some special case of the payoff matrix discovered in our paper [1] in which our macroscopic quantum game is totally defined. This case was called by us the case of "eigenequilibrium".

In quantum game on the quadrangle [1] with payoff matrix (see tabl.1) the average payoff of Alice can be written as

$$\langle H \rangle = \frac{1}{4} (g(x, y) + \text{tr } C)$$

where

$$g(x, y) = -\langle x, Ay \rangle + \langle x, M_\theta^\dagger \omega \rangle - \langle M_\tau^\dagger \omega, y \rangle,$$

$A = M_\theta^\dagger C M_\tau$, and x, y unit vectors on the plane: $|x| = 1, |y| = 1$. Here

$$M_\varphi = \begin{bmatrix} \cos \varphi & -\sin \varphi \\ \sin \varphi & \cos \varphi \end{bmatrix}, \quad n = c_1 + c_3, \quad m = c_2 + c_4$$

$$\omega = \begin{bmatrix} c_3 - c_1 \\ c_4 - c_2 \end{bmatrix}, \quad C = \begin{bmatrix} n & 0 \\ 0 & m \end{bmatrix}$$

An equilibrium (x, y) is called *eigenequilibrium*, if it is an eigenvector of the matrix

$$\mathcal{A} = \begin{bmatrix} O & A \\ A^\dagger & O \end{bmatrix}$$

The following proposition proved in [7].

Proposition 1. *If the eigenequilibrium exists, then ω is a common eigenvector of the matrices $CM_\theta M_\theta^\dagger$ $CM_\tau M_\tau^\dagger$.*

A game is said to be *non-degenerate*, if

$$\Delta = \begin{vmatrix} n & m \\ \omega_1^2 & \omega_2^2 \end{vmatrix} \neq 0$$

Proposition 2. *If the game is non-degenerate, then the necessary condition for the eigenequilibrium to exist is the coincidence of the angular parameters $\theta = \tau$. In this case their values are completely determined by the payoff coefficients of the game $\{c_j\}$:*

$$\cos 2\theta = \cos 2\tau = \frac{(m - n)\omega_1\omega_2}{\Delta} \quad (9)$$

Further finding *eigenequilibrium* of *non-degenerate* games, calculate θ using (9) and put $M = M_\theta$, $z = M^\dagger \omega$. In this case $A = A^\dagger = M^\dagger CM$ and the matrix A non-negatively defined.

Proposition 3. (EXISTENCE THEOREM) *Let a vector ω be an eigenvector of the matrix CMM^\dagger and $\langle Az, z \rangle \leq |z|^3$. Then the strategies $x = y = z/|z|$ form an eigenequilibrium.*

Proposition 4. (MULTIPLE NASH-EQUILIBRIUM) *Let a vector ω be an eigenvector of the matrix CMM^\dagger and $\langle Az, z \rangle = |z|^3$. Then there are two eigenequilibrium $x = y = z/|z|$ $x = -z/|z|$, $y = z/|z|$.*

Proposition 5. (UNIQUENESS THEOREM) *Let there is a game with a non-degenerate equilibrium $\langle Az, z \rangle \neq |z|^3$. Then all possible equilibrium are exhausted by it.*

So, it occurs that optimal strategies of the players are defined not so by the representation of the ortholattice as by the ortholattice itself and by the payoff structure of the game.

For this case as in quantum Stern-Gerlach quantum game the angles are prescribed by the rule of the game and the only choices for Alice and

Bob training their fireflies concern probability distributions satisfying the constraint with this angle.

The optimal choice corresponds to Nash equilibrium existing for this angle. For other choice of the angle Nash equilibrium does not exist and clever Alice and Bob will not use them at all.

4. Example of the multiple quantum Nash-equilibrium

For

$$c_1 = 1; \quad c_2 = 2; \quad c_3 = 99; \quad c_4 = 98$$

the angles is equal $\theta = \tau = 45^\circ$ and the optimal strategies of Alice and Bob are

$$p_1 = q_1 \approx 0,857; \quad p_2 = q_2 \approx 0,622; \quad \langle H \rangle = 50$$

For

$$c_1 = 1; \quad c_2 = 2; \quad c_3 = 9; \quad c_4 = 8$$

the angles is equal $\theta = \tau = 45^\circ$ and there are *two* eigenequilibrium.

First equilibrium:

$$p_1 = q_1 = 0,9; \quad p_2 = q_2 = 0,8; \quad \langle H \rangle = 5$$

Second equilibrium:

$$p_1 = 0,1; \quad q_1 = 0,9; \quad p_2 = 0,2; \quad q_2 = 0,8; \quad \langle H \rangle = 5$$

5. Quantum equilibrium against the classical one

If one considers the classical game with the same payoff matrix one can obtain [1] for the average profit:

$$h = (c_1^{-1} + c_3^{-1})^{-1} + (c_2^{-1} + c_4^{-1})^{-1}$$

It can be shown in some cases that the quantum equilibrium is more profitable than the classical one. Consider the following example: $c_1 = 1$, $c_2 = 9$,

$c_3 = 10$, $c_4 = 2$ the optimal strategies of Alice and Bob are

$$p_1 = q_1 = \frac{130 + 9\sqrt{130}}{260} \approx 0.895, \quad p_2 = q_2 = \frac{130 - 7\sqrt{130}}{260} \approx 0.193$$

The optimal profit in the classical game is smaller than in the quantum one: $\langle h \rangle = 28/11$, $\langle H \rangle = 11/4$.

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